JOURNAL OF APPROXIMATION THEORY 59, 296–299 (1989)

Best Approximation in $L^p(\mu, X)$, II

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Communicated by E. W. Cheney

Received June 8, 1987

The object of this paper is to prove the following theorem: If Y is a closed subspace of the Banach space X, then $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$ if and only if $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$ for every p, $1 < p < \infty$. As an application of this result we prove that if Y is either reflexive or Y is a separable proximinal dual space, then $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$. © 1989 Academic Press, Inc.

INTRODUCTION

Let (Ω, μ) be a finite measure space. The space of Bochner p-integrable functions defined on (Ω, μ) with values in a Banach space X is denoted by $L^p(\mu, X)$. It is well known [1] that $L^p(\mu, X)$ is a Banach space under the norm

$$\|f\|_p = \left(\int \|f(t)\|^p \, d\mu(t)\right)^{1/p}, \qquad 1 \leqslant p < \infty.$$

A subspace E in a Banach space F is said to be proximinal if for each $x \in F$ there is at least one $y \in E$ such that

$$\|x - y\| = d(x, E) = \inf\{\|x - z\| : z \in E\}.$$

The element y is called a best approximant of x in E.

In [3], Light and Cheney proved that if Y is a finite-dimensional subspace of the Banach space X, then $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$. In [2], Khalil proved that $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$ if Y is reflexive. In this paper we prove that $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$ if and only if $L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$, $1 < p < \infty$. As a consequence, the result in [2] follows immediately. Further, if Y is a separable proximinal dual space then $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$.

Throughout this paper, if X is a Banach space, then X^* denotes the dual of X. If Y is a subspace of X, we set $Y^\perp = \{x^* \in X^* : x^*(y) = 0 \text{ for all } y \in T\}$. The set of real numbers is denoted by R.

All Banach space in this paper are assumed to be real Banach spaces.

0021-9045/89 $3.00

I. Proximinality Relations in $L^p(\mu, X)$

Let X be a Banach space and let Y be a closed subspace of X. The following is the main result of this paper:

Theorem 1.1. *Let $1 < p < \infty$. The following are equivalent:*

(i) *$L^p(\mu, Y)$ is proximinal in $L^p(\mu, X)$*

(ii) *$L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$.*

Proof. (ii) $\to$ (iii). Let $f \in L^p(\mu, X)$. Since the measure space (Ω, μ) is finite, $f \in L^1(\mu, X)$. By assumption, there exists $g \in L^1(\mu, Y)$ such that $\|f - g\|_1 \leqslant \|f - R\|_1$ for all $h \in L^1(\mu, Y)$. By Lemma 2.10 of [4],

$$\|f(t) - g(t)\| \leqslant \|f(t) - y\|$$

μ-almost everywhere and for all $y \in Y$. Hence

$$\|f(t) - g(t)\| \leqslant \|f(t) - w(t)\|$$

μ-almost everywhere for all $w \in L^p(\mu, Y)$. Since $0 \in Y$, it follows that $\|g(t)\| \leqslant 2\|f(t)\|$. Hence $g \in L^p(\mu, Y)$, and

$$\|f - g\|_p \leqslant \|f - w\|_p$$

for all $w \in L^p(\mu, Y)$.

Conversely. (i) $\to$ (ii). Consider the map

$$J\colon\ L^1(\mu, X) \to L^p(\mu, X)$$
$$J(f)(t) = \|f(t)\|^{1/p - 1} f(t)$$

if $f(t) \neq 0$, and $J(f)(t) = 0$ otherwise. Then

$$\|J(f)(t)\| = \|f(t)\|^{1/p}.$$

Hence $\|J(f)\|_p^p = \|f\|_1$. Clearly J is $(1-1)$. Further, if $g \in L^p(\mu, X)$, then $f(t) = \|g(t)\|^{p-1} g(t) \in X$ and $\|f(t)\| = \|g(t)\|^p$. Thus $f \in L^1(\mu, X)$. Further

$$\begin{aligned} J(f)(t) &= [\|f(t)\|]^{1/p-1} \cdot \|g(t)\|^{p-1} g(t) \\ &= \|g(t)\|^{1-p} \cdot \|g(t)\|^{p-1} g(t) = g(t). \end{aligned}$$

Hence J is onto. Also $J(L^1(\mu, Y)) = L^p(\mu, Y)$.

Now, let $f \in L^1(\mu, X)$. With no loss of generality we can assume that $f(t) \neq 0$ μ-almost everywhere, for otherwise we can restrict our measure to the support of f. Since $J(f) \in L^p(\mu, X)$, then by assumption (ii), there exists some $g \in L^1(\mu, Y)$ such that

$$\|J(f) - J(g)\|_p \leqslant \|J(f) - J(h)\|_p$$

for all $h \in L^1(\mu, Y)$. Using the same argument as in Lemma 2.10 of [4], we get

$$\|J(f)(t) - J(g)(t)\| \leqslant \|J(f)(t) - y\|$$

μ-almost everywhere for all $y \in Y$. Hence

$$\|J(f)(t) - J(g)(t)\| \leqslant \|J(f)(t) - \|f(t)\|^{1/p-1} y\|,$$

μ-almost everywhere for all $y \in Y$. Multiplying both sides of the last inequality by $\|f(t)\|^{1-1/p}$ we get

$$\|f(t) - \|f(t)\|^{1-1/p} \cdot \|g(t)\| \; g(t)\| \leqslant \|f(t) - y\|$$

for all $y \in Y$. Set $w(t) = \|f(t)\|^{1-1/p} \; \|g(t)\|^{1/p-1} g(t)$. Since $g(t)$ is a best approximant of $f(t)$ in Y, and $0 \in y$, it follows that $\|g(t)\| \leqslant 2\|f(t)\|$. Hence $w \in L^1(\mu, Y)$. Consequently

$$\|f(t) - w(t)\| \leqslant \|f(t) - \theta(t)\|$$

μ-almost everywhere for all $\theta \in L^1(\mu, Y)$, and so g is a best approximant of f in $L^1(\mu, Y)$. This ends the proof of the theorem.

As a corollary to Theorem 1.1, we prove

THEOREM 1.2 [Khalil [2]]. *If Y is a reflexive subspace of X, then $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$.*

Proof. The subspace $L^2(\mu, Y)$ is reflexive in $L^2(\mu, X)$. Hence proximinal. Theorem 1.1 implies that $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$. This ends the proof. Q.E.D.

THEOREM 1.3. *Let $Y = Z^* \subseteq X$. Then $L^1(\mu, Y)$ is proximinal in $L^1(\mu, X)$ if Y is separable and proximinal in X.*

Proof. Let $F \in L^2(\mu, X)$ and let X_1 be the smallest separable closed subspace of X that contains the range of F. Let (x_n) be a countable dense subset of X_1 and let F_n be a sequence of simple functions such that $\|F_n - F\|_2 \to 0$. We can choose each F_n to have values in $\{x_1, x_2, ...\}$.

Since Y is proximinal each F_n has a best approximant $\tilde{F}_n$ in $L^2(\mu, Y)$. In fact each F_n is simple and $F_n(t)$ is the best approximant of $F(t)$ in Y μ a.e. [2]. Let Y_1 be the smallest closed subspace of Y that contains the range of F_n for all n. So Y_1 is separable. We can assume that $Y_1 = Y$. Let (y_n) be a countable dense subset in Y_1. Set

$$Q = \{y_1, y_2, ...\} \cup \{x_i : x_i \notin Y_1\}.$$

Then Q is dense in x_1 and $Q \cap Y_1$ is dense in Y. Further, one can use the Hahn–Banach theorem to define a sequence of linear functionals (z_i^*) in X_1^* such that $\|Z_i^*\| = 1$ and $\langle z_i^*, z_i \rangle = \|z_i\|$. Since (z_i) is dense in X_1 and (Z_i^*) is norming for (z_i) $(\|z_i\| = \sup_i |\langle z_i^*, z_i \rangle|)$, then (z_i^*) is norming for X_1 (and for Y). Further (z_i^*) is total for X_1 (and for Y_1) [4, p. 24].

Let $E = \{h \otimes z_i^* \colon h \in L^2(\mu), \ i = 1, 2, 3, ...\}$. Then the Hahn–Banach theorem and the totality of (z_i^*) imply that E is total for $L^2(\mu, X_1)$. Further, the density of the simple function (with range in Q) in $L^2(\mu, X_1)$ implies that $S(E) = \text{span}(E)$ is norming for $L^2(\mu, X_1)$.

Since $L^2(\mu)$ is reflexive, we use the Cantor diagonalization process to have a subsequence of $(\tilde{F}_n)$, say $(\tilde{F}_{n_k})$, such that $\lim_{n_k} \langle \tilde{F}_{n_k}, \phi \rangle$ exists for all $\phi \in S(E)$. Let $\tilde{F}$ be the linear functional on $L^2(\mu, Y^*)$ defined by $\langle \tilde{F}, \phi \rangle = \lim_{n_k} \langle \tilde{F}_{n_k}, \phi \rangle$ for all $\phi \in S(E)$. The Hahn–Banach theorem can be used to ensure that $\tilde{F} \in [L^2(\mu, Y^*)]^*$. The fact that Y is a dual space and $\|\tilde{F}_n(t)\| \leqslant 2\|F_n(t)\|$ μ a.e. implies that $\tilde{F} \in L^2(\mu, Y)$ [4, p. 91]. Now let $\phi \in S(E)$, $\|\phi\| \leqslant 1$, and $\varepsilon > 0$ be given. Then

$$\begin{aligned} |\langle F - \tilde{F}, \phi \rangle| &\leqslant |\langle F - F_n, \phi \rangle| + |\langle F_n - \tilde{F}_n, \phi \rangle| + |\langle \tilde{F}_n - \tilde{F}, \phi \rangle| \\ &\leqslant \|\langle F - F_n\|_2 + \|F_n - F_n\|_1 + |\langle F_n - F, \phi \rangle|. \end{aligned}$$

By choosing n large enough one gets

$$|\langle F - F, \phi \rangle| \leqslant 2\varepsilon + \|F_n - V\|_2 \leqslant 2\varepsilon + \|F - V\|$$

for all $V \in L^2(\mu, Y)$. Since ε is arbitrary and $S(E)$ is norming for $L^2(\mu, X_1)$, using Theorem 1.1, the result follows. Q.E.D.

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